Department of Mathematics, K.T.H.M. College, Nashik
T.Y.B.Sc. - Problems in Real Analysis - I

Chapter 2: Sequence of Real Numbers

1. If $L \in \mathbb{R}$, $L < M + \epsilon$ for every $\epsilon > 0$, prove $L \leq M$.

   Ans: Suppose on contrary that $L > M$. Take $\epsilon = L - M > 0$. By hypothesis,

   $L < M + \epsilon$

   $\implies L < M + (L - M)$

   $\implies L < L$

   This is a contradiction. Hence $L \leq M$.

2. If $L \in \mathbb{R}$, $L \leq M + \epsilon$ for every $\epsilon > 0$, prove $L \leq M$.

   Ans: Suppose on contrary that $L > M$. Take $\epsilon = \frac{L-M}{2} > 0$. By hypothesis,

   $L < M + \epsilon$

   $\implies L < M + \frac{L-M}{2}$

   $\implies L - M < \frac{L-M}{2}$

   $\implies 1 < \frac{1}{2}$

   This is a contradiction. Hence $L \leq M$.

Archimedian Property of Real Numbers: For any real number x, there is a positive integer N such that $N > x$.

3. Prove that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

   Ans: Let $\epsilon > 0$. By Archimedian property of real numbers, there is $N \in I$ such that

   $$N > \frac{1}{\epsilon}$$

   Consider

   $n \geq N$

   $\implies n > \frac{1}{\epsilon}$

   $\implies \frac{1}{\epsilon} < n$

   $\implies \frac{1}{n} < \epsilon$

   $\implies |\frac{1}{n} - 0| < \epsilon$

   Thus for $N > \frac{1}{\epsilon}$,

$n \geq N \implies |\frac{1}{n} - 0| < \epsilon$

This proves that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

4. Prove that $\lim\limits_{n\to\infty} \frac{1}{n+3} = 0$.

   Ans: Let $\epsilon > 0$. By Archimedian property of real numbers, there is $N \in I$ such that

   $$N > \frac{1}{\epsilon}$$

   Consider

   $n \geq N$

   $\implies n > \frac{1}{\epsilon}$

   $\implies \frac{1}{\epsilon} < n$

   $\implies \frac{1}{n} < \epsilon$

   $\implies \frac{1}{n+3} < \epsilon \qquad (\because \frac{1}{n+3} < \frac{1}{n})$

   $\implies |\frac{1}{n+3} - 0| < \epsilon$

   Thus for $N > \frac{1}{\epsilon}$,

   $n \geq N \implies |\frac{1}{n+3} - 0| < \epsilon$

   This proves that $\lim\limits_{n\to\infty} \frac{1}{n+3} = 0$

5. Prove that $\lim\limits_{n\to\infty} \frac{1}{n-3} = 0$.

   Ans: Let $\epsilon > 0$. By Archimedian property of real numbers, there is $N(\geq 3) \in I$ such that

   $$N > \frac{1}{\epsilon} + 3$$

   Consider

   $n \geq N$

   $\implies n > \frac{1}{\epsilon} + 3$

   $\implies \frac{1}{\epsilon} + 3 < n$

   $\implies \frac{1}{\epsilon} < n - 3$

   $\implies \frac{1}{n-3} < \epsilon$

$\implies |\frac{1}{n-3} - 0| < \epsilon$

Thus for $N > \frac{1}{\epsilon} + 3$,

$n \geq N \implies |\frac{1}{n-3} - 0| < \epsilon$

This proves that $\lim\limits_{n\to\infty} \frac{1}{n-3} = 0$

6. Prove that $\lim\limits_{n\to\infty} \frac{10^7}{n} = 0$.

   Ans: Let $\epsilon > 0$. By Archimedian property of real numbers, there is $N \in I$ such that

   $$N > \frac{10^7}{\epsilon}$$

   Consider

   $n \geq N$

   $\implies n > \frac{10^7}{\epsilon}$

   $\implies \frac{10^7}{\epsilon} < n$

   $\implies \frac{10^7}{n} < \epsilon$

   $\implies |\frac{10^7}{n} - 0| < \epsilon$

   Thus for $N > \frac{10^7}{\epsilon}$,

   $n \geq N \implies |\frac{10^7}{n} - 0| < \epsilon$

   This proves that $\lim\limits_{n\to\infty} \frac{10^7}{n} = 0$

7. $\{s_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ is divergent.

   Ans: Suppose on contrary that $\{s_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ is convergent and let $\lim\limits_{n\to\infty} s_n = L$. Then for $\epsilon = 1 > 0$, there is $N \in I$ such that

   $n \geq N \implies |s_n - L| < 1$

   $\implies (n \geq N \implies |n - L| < 1)$

   $\implies (n \geq N \implies L - 1 < n < L + 1)$

   $\implies L - 1 < N, N + 2 < L + 1$

   $\implies (N + 2) - N < (L + 1) - (L - 1)$

   $\implies 2 < 2$

This is a contradiction. Hence $\{s_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ is divergent.

8. $\{s_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ is divergent.

   Ans: Suppose on contrary that $\{s_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ is convergent and let $\lim\limits_{n\to\infty} s_n = L$. Then for $\epsilon = 1 > 0$, there is $N \in I$ such that

   $n \geq N \implies |s_n - L| < 1$

   $\implies (n \geq N \implies |(-1)^n - L| < 1)$

   $\implies (n \geq N \implies L - 1 < (-1)^n < L + 1)$

   $\implies L - 1 < (-1)^{2N+1}, (-1)^{2N} < L + 1$

   $\implies L - 1 < -1, 1 < L + 1$

   $\implies 1 - (-1) < (L + 1) - (L - 1)$

   $\implies 2 < 2$

   This is a contradiction. Hence $\{s_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ is divergent.

9. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if, for every $\epsilon > 0$,

   $$n \geq N \implies |s_n - L| < \epsilon$$

   where N does *not* depend on ϵ, prove that all but a finite number of terms of $\{s_n\}_{n=1}^{\infty}$ are equal to L.

   Ans: We shall use the following result

   "If $L \in \mathbb{R}$, $L \leq M + \epsilon$ for every $\epsilon > 0$, then $L \leq M$." $\ldots(*)$

   Consider

   $n \geq N \implies |s_n - L| < \epsilon$ for every $\epsilon > 0$

   $\implies (n \geq N \implies L - \epsilon < s_n < L + \epsilon$ for every $\epsilon > 0)$

   $\implies (n \geq N \implies L < s_n + \epsilon$ and $s_n < L + \epsilon$ for every $\epsilon > 0)$

   $\implies (n \geq N \implies L \leq s_n$ and $s_n \leq L)$ (by (*))

   $\implies (n \geq N \implies s_n = L)$

   $\implies s_N = s_{N+1} = s_{N+2} = \cdots = L$

$\implies$ all but a finite number of terms of $\{s_n\}_{n=1}^{\infty}$ are equal to L.

10. (a) Find $N \in I$ such that

$$n \geq N \implies |\tfrac{2n}{n+3} - 2| < \tfrac{1}{5}$$

(b) Prove that $\lim\limits_{n\to\infty} \frac{2n}{n+3} = 2$

Ans: (a) : Take $N = 31$. Consider

$n \geq N$

$\implies n \geq 31$

$\implies n > 30$

$\implies 30 < n$

$\implies \frac{6}{n} < \frac{1}{5}$

$\implies \frac{6}{n+3} < \frac{1}{5} \qquad (\because \frac{6}{n+3} < \frac{6}{n})$

$\implies |\frac{2n}{n+3} - 2| < \frac{1}{5}$

Thus for $N = 30$

$n \geq N \implies |\frac{2n}{n+3} - 2| < \frac{1}{5}$

(b) Let $\epsilon > 0$. By Archimedian property of real numbers, there is $N \in I$ such that

$$N > \tfrac{6}{\epsilon}$$

Consider

$n \geq N$

$\implies n > \frac{6}{\epsilon}$

$\implies \frac{6}{\epsilon} < n$

$\implies \frac{6}{n} < \epsilon$

$\implies \frac{6}{n+3} < \epsilon \qquad (\because \frac{6}{n+3} < \frac{6}{n})$

$\implies |\frac{2n}{n+3} - 2| < \epsilon$

Thus for $N > \frac{1}{\epsilon}$,

$n \geq N \implies |\frac{2n}{n+3} - 2| < \epsilon$

This proves that $\lim\limits_{n\to\infty} \frac{2n}{n+3} = 2$

11. (a) Find $N \in I$ such that

$$n \geq N \implies \frac{1}{\sqrt{n+1}} < 0.03$$

(b) Prove that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

Ans: **(a)** By Archimedian property of real numbers, there is $N \in I$ such that

$$N > \frac{10000}{9}$$

Consider

$n \geq N$

$\implies n > \frac{10000}{9}$

$\implies \frac{10000}{9} < n$

$\implies \frac{1}{n} < \frac{9}{10000}$

$\implies \frac{1}{n+1} < \frac{9}{10000} \qquad (\because \frac{1}{n+1} < \frac{1}{n})$

$\implies \frac{1}{\sqrt{n+1}} < \frac{3}{100}$

$\implies \frac{1}{\sqrt{n+1}} < 0.03$

Thus for $N > \frac{10000}{9}$

$n \geq N \implies \frac{1}{\sqrt{n+1}} < 0.03$

(b) By Archimedian property of real numbers, there is $N \in I$ such that

$$N > \frac{1}{\epsilon^2}$$

Consider

$n \geq N$

$\implies n > \frac{1}{\epsilon^2}$

$\implies \frac{1}{\epsilon^2} < n$

$\implies \frac{1}{n} < \epsilon^2$

$\implies \frac{1}{n+1} < \epsilon^2 \qquad (\because \frac{1}{n+1} < \frac{1}{n})$

$\implies \frac{1}{\sqrt{n+1}} < \epsilon$

$\implies \frac{1}{\sqrt{n+1}} < \epsilon$

$\implies |\frac{1}{\sqrt{n+1}} - 0| < \epsilon$

Thus for $N > \frac{1}{\epsilon^2}$

$n \geq N \implies |\frac{1}{\sqrt{n+1}} - 0| < \epsilon$

This proves that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.

12. Suppose $\{s_n\}_{n=1}^{\infty}$ is a sequence of positive numbers and $0 < x < 1$. If $s_{n+1} < xs_n$ for all $n \in I$, prove $\lim\limits_{n\to\infty} s_n = 0$.

Ans: $0 < x < 1$

$\implies 0 < xs_n < s_n \qquad (\because s_n > 0)$

$\implies 0 < s_{n+1} < s_n \qquad (\because s_{n+1} < xs_n)$

Thus the sequence $\{s_n\}_{n=1}^{\infty}$ is non-increasing and bounded below by 0. So it is convergent. Take $\lim\limits_{n\to\infty} s_n = L$. Consider

$0 < s_{n+1} < xs_n$

$\implies 0 < s_{n+1} - xs_n < 0$

$\implies 0 \leq \lim\limits_{n\to\infty} (s_{n+1} - xs_n) \leq 0$

$\implies \lim\limits_{n\to\infty} (s_{n+1} - xs_n) = 0$

$\implies \lim\limits_{n\to\infty} s_{n+1} - x \lim\limits_{n\to\infty} s_n = 0$

$\implies L - xL = 0 \qquad (\because \lim\limits_{n\to\infty} s_{n+1} = L$ as $\{s_{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty})$

$\implies (1-x)L = 0$

$\implies L = 0 \qquad (\because x - 1 \neq 0$ as $x \neq 1)$

13. Suppose $\lim\limits_{n\to\infty} \frac{s_n - 1}{s_n + 1} = 0$. Prove that $\lim\limits_{n\to\infty} s_n = 1$.

Ans: Take $\epsilon_n = \frac{s_n - 1}{s_n + 1}$. Then

$\implies \epsilon_n(s_n + 1) = s_n - 1$

$\implies 1 + \epsilon_n = s_n(1 - \epsilon_n)$

$\implies s_n = \frac{1+\epsilon_n}{1-\epsilon_n}$

$\implies \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1+\epsilon_n}{1-\epsilon_n}$

$\implies \lim\limits_{n\to\infty} s_n = \frac{1+\lim\limits_{n\to\infty} \epsilon_n}{1-\lim\limits_{n\to\infty} \epsilon_n}$

$\implies \lim\limits_{n\to\infty} s_n = \frac{1+0}{1-0}$ $\quad (\because \lim\limits_{n\to\infty} \epsilon_n = \lim\limits_{n\to\infty} \frac{s_n-1}{s_n+1} = 0)$

$\implies \lim\limits_{n\to\infty} s_n = 1$

14. If $s_n = \frac{10^n}{n!}$, find $N \in I$ such that $n \geq N \implies s_{n+1} < s_n$.

Ans: Take $N = 10$. Consider

$n \geq N$

$\implies n > 9$

$\implies 10 < n + 1$

$\implies \frac{10^{n+1}}{10^n} < \frac{(n+1)!}{n!}$

$\implies \frac{10^{n+1}}{(n+1)!} < \frac{10^n}{n!}$

$\implies s_{n+1} < s_n$

Thus for $N = 10$, $n \geq N \implies s_{n+1} < s_n$.

15. For $n \in I$, let $s_n = \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}$. Prove that $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} s_n \leq \frac{1}{2}$.

Ans: Consider

$1 < 2$

$\implies 2n + 1 < 2n + 2$

$\implies \frac{2n+1}{2n+2} < 1$

$\implies \frac{1.3.5.\cdots(2n-1)(2n+1)}{2.4.6.\cdots 2n(2n+2)} \times \frac{2.4.6.\cdots(2n)}{1.3.5.\cdots(2n-1)} < 1$

$\implies \frac{1.3.5.\cdots(2n-1)(2n+1)}{2.4.6.\cdots 2n(2n+2)} < \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}$

$\implies s_{n+1} < s_n$

This is true for all $n \in I$. Therefore $\{s_n\}_{n=1}^{\infty}$ is non-increasing. Also it is lower bounded by 0. So it is convergent. Finally,

$s_n \leq s_1$

$\implies s_n \leq \frac{1}{2}$

$\implies \lim\limits_{n\to\infty} s_n \leq \frac{1}{2}$

16. For $n \in I$ let $s_n = \frac{2.4.6\cdots(2n)}{1.3.5\cdots(2n-1)}\frac{1}{n^2}$. Prove that $\{s_n\}_{n=1}^{\infty}$ is non-increasing.

    Ans: Consider

    $\frac{s_n}{s_{n+1}} = \frac{2.4.6\cdots(2n)}{1.3.5\cdots(2n-1)} \times \frac{1.3.5\cdots(2n-1)(2n+1)}{2.4.6\cdots(2n)(2n+2)} \times \frac{(n+1)^2}{n^2}$

    $= \frac{(2n+1)(n+1)^2}{(2n+2)n^2}$

    $= \frac{(2n+1)(n^2+2n+1)}{2n^3+2n^2}$

    $= \frac{2n^3+5n^2+4n+1}{2n^3+2n^2}$

    > 1

    Thus

    $\frac{s_n}{s_{n+1}} > 1$

    $\implies s_n > s_n + 1$

    This is true for all $n \in I$. So $\{s_n\}_{n=1}^{\infty}$ is non-increasing.

17. For $n \in I$, let $t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Prove that $\{t_n\}_{n=1}^{\infty}$ is non-decreasing, bounded and hence convergent.

    Ans: **A) $\{t_n\}_{n=1}^{\infty}$ is non -decreasing:** Consider

    $\frac{1}{(n+1)!} \geq 0$

    $\implies t_n + \frac{1}{(n+1)!} \geq t_n$

    $\implies t_{n+1} \geq t_n$

    This holds for all $n \in I$. So $\{t_n\}_{n=1}^{\infty}$ is non-decreasing.

    B) $\{t_n\}_{n=1}^{\infty}$ is bounded: Clearly $\{t_n\}_{n=1}^{\infty}$ is lower bounded by 1. Consider

    $n \geq 2$

    $\implies 1.2.3\ldots n \geq \underbrace{2.2\ldots2}_{(n-1)\ terms}$

$\implies n! \geq 2^{n-1}$

Thus

$n \geq 2 \implies n! \geq 2^{n-1}$

so that

$n > 2 \implies \frac{1}{n!} \leq \frac{1}{2^{n-1}}$

Now,

t_n

$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$

$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$

$= 1 + \frac{1.(1-2^{-n})}{1-2^{-1}}$

$= 1 + 2(1-2^{-n})$

$\leq 1 + 2$

$= 3$

So $\{t_n\}_{n=1}^{\infty}$ is upper bounded by 3.

Thus $\{t_n\}_{n=1}^{\infty}$ is monotone and bounded. So it is convergent.

18. Show that the sequence $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ is convergent.

Ans: Consider

s_n

$= (1+\frac{1}{n})^n$

$= \sum\limits_{k=0}^{n} {}^nC_k \frac{1}{n^k}$

$= \sum\limits_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-(k-1))}{1.2.3.\cdots k} \frac{1}{n^k}$

$= \sum\limits_{k=0}^{n} \frac{1}{1.2.3.\cdots k}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})$

$\leq \sum\limits_{k=0}^{n} \frac{1}{1.2.3.\cdots k}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{k-1}{n+1})$

$\leq \sum_{k=0}^{n+1} \frac{1}{1.2.3.\cdots k}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{k-1}{n+1})$

$= \sum_{k=0}^{n+1} \frac{(n+1)((n+1)-1)((n+1)-2)\cdots((n+1)-(k-1))}{1.2.3.\cdots k}\frac{1}{(n+1)^k}$

$= \sum_{k=0}^{n+1} {}^{(n+1)}C_k \frac{1}{(n+1)^k}$

$= s_{n+1}$

Thus $s_n \leq s_{n+1}$. Therefore

$\{s_n\}_{n=1}^{\infty}$ is nondecreasing. $\ldots(I)$

Also

s_n

$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{1.2}.\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 1}{1.2\cdots n}\frac{1}{n^n}$

$< 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.\cdots n}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

$= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$

$< 1 + \frac{1}{1-\frac{1}{2}}$

$= 3$

Thus $s_n < 3$ for all $n \in I$. Therefore

$\{s_n\}_{n=1}^{\infty}$ is upper bounded by 3 $\ldots(II)$

From (I) and (II), $\{s_n\}_{n=1}^{\infty}$ is nondecreasing and upper bounded. Hence $\{s_n\}_{n=1}^{\infty}$ is convergent.

19. For any $a, b \in \mathbb{R}$ show that

    $||a| - |b|| \leq |a-b|$.

    Then prove that $\{|s_n|\}_{n=1}^{\infty}$ converges to $|L|$ if $\{s_n\}_{n=1}^{\infty}$ converges to L.

    Ans:

    $|a| = |(a-b)+b| \leq |a-b| + |b|$

    $\implies |a| - |b| \leq |a-b|$ ...(I)

and

$|b| = |(b-a)+b| \leq |b-a| + |a|$

$\implies |b| - |a| \leq |b-a|$

$\implies -(|a|-|b|) \leq |a-b|$ ...(II)

From (I) and (II), $||a|-|b|| \leq |a-b|$. Next, Consider

$\{s_n\}_{n=1}^{\infty}$ converges to L

$\implies$ for every $\epsilon > 0$, there exists $N \in I$ such that $(n \geq N \implies |s_n - L| < \epsilon)$

$\implies$ for every $\epsilon > 0$, there exists $N \in I$ such that $(n \geq N \implies ||s_n| - |L|| < \epsilon)$

$(\because ||s_n| - |L|| \leq |s_n - L|)$

$\implies \{|s_n|\}_{n=1}^{\infty}$ converges to $|L|$

20. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\lim\limits_{n\to\infty} s_{2n} = \lim\limits_{n\to\infty} s_{2n-1} = L$, prove that $\lim\limits_{n\to\infty} s_n = L$

Ans: Take $\epsilon > 0$.Since $\lim\limits_{n\to\infty} s_{2n} = L$, there is $N_1 \in I$ such that

$n > 2N_1 \implies |s_{2n} - L| < \epsilon$ $\ldots(I)$

Similarly as $\lim\limits_{n\to\infty} s_{2n-1} = L$, there is $N_2 \in I$ such that

$n > 2N_2 \implies |s_{2n-1} - L| < \epsilon$ $\ldots(II)$

Take $N' = \max\{N_1, N_2\}$. Then from (I) and (II)

$n \geq N' \implies |s_{2n} - L| < \epsilon$ and $|s_{2n-1} - L| < \epsilon$

$\implies |s_{2N'-1} - L| < \epsilon, |s_{2N'} - L| < \epsilon, |s_{2N'+1} - L| < \epsilon, |s_{2N'+2} - L| < \epsilon, \ldots$

$\implies |s_N - L| < \epsilon, |s_{N+1} - L| < \epsilon, |s_{N+2} - L| < \epsilon, |s_{N+3} - L| < \epsilon, \ldots$ where $N = 2N' - 1$

$\implies (n \geq N \implies |s_n - L| < \epsilon)$

This proves that $\lim\limits_{n\to\infty} s_n = L$

21. Give an example of a sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers for which $\{|s_n|\}_{n=1}^{\infty}$ converges but $\{s_n\}_{n=1}^{\infty}$ does not.

Ans: $\{s_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$. The sequence $\{s_n\}_{n=1}^{\infty}$ oscillates whereas $\{|s_n|\}_{n=1}^{\infty}$ converges to 1.

22. Prove that if $\{|s_n|\}_{n=1}^{\infty}$ converges to 0, then $\{s_n\}_{n=1}^{\infty}$ converges to 0.

    Ans:

    $\{|s_n|\}_{n=1}^{\infty}$ converges to 0

    $\implies$ for every $\epsilon > 0$, there exists $N \in I$ such that $(n \geq N \implies ||s_n| - 0| < \epsilon)$

    $\implies$ for every $\epsilon > 0$, there exists $N \in I$ such that $(n \geq N \implies |s_n| < \epsilon)$

    $\implies$ for every $\epsilon > 0$, there exists $N \in I$ such that $(n \geq N \implies |s_n - 0| < \epsilon)$

    $\implies \{s_n\}_{n=1}^{\infty}$ converges to 0

23. Prove that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to infinity.

    Ans: Let $M > 0$. By Archimedian property of real numbers, there is $N \in I$, such that

    $$N > M^2 \qquad \ldots \text{(I)}$$

    Consider

    $n \geq N$

    $\implies n > M^2$ (from (I))

    $\implies \sqrt{n} > M$

    Thus

    $n \geq N \implies \sqrt{n} > M$

    This proves that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to infinity.

24. Prove that $\{\log n\}_{n=1}^{\infty}$ diverges to infinity.

    Ans: Let $M > 0$. By Archimedian property of real numbers, there is $N \in I$, such that

    $$N > e^M \qquad \ldots \text{(I)}$$

    Consider

    $n \geq N$

    $\implies n > e^M$ (from (I))

    $\implies \log n > \log(e^M)$ $(\because f(x) = \log(x)$ is increasing function)

    $\implies \log n > M$

Thus

$n \geq N \implies \log n > M$

This proves that $\{\log n\}_{n=1}^{\infty}$ diverges to infinity.

25. Prove that $\{\log \frac{1}{n}\}_{n=1}^{\infty}$ diverges to minus infinity.

Ans: Let $M > 0$. By Archimedian property of real numbers, there is $N \in I$, such that

$$N > e^{M} \qquad \ldots (\mathrm{I})$$

Consider

$n \geq N$

$\implies n > e^{M}$ (from (I))

$\implies \log n > \log(e^{M})$ ($\because f(x) = \log(x)$ is increasing function)

$\implies \log n > M$

$\implies -\log n < -M$

$\implies log\frac{1}{n} < -M$

Thus

$n \geq N \implies \log \frac{1}{n} < -M$

This proves that $\{\log \frac{1}{n}\}_{n=1}^{\infty}$ diverges to minus infinity.

26. Prove that $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$ is convergent.

Ans: $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$

$= \lim_{n\to\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) (\sqrt{n+1} + \sqrt{n})$

$= \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1} + \sqrt{n}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$= 0$

So the sequence $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$ converges to 0.

27. Prove that if the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ diverges to infinity, then $\{-s_n\}_{n=1}^{\infty}$ diverges to minus infinity.

Ans: Let $M > 0$. Since $\{s_n\}_{n=1}^{\infty}$ diverges to infinity, there is $N \in I$ such that

$$n \geq N \implies s_n > M$$

implying that

$$n \geq N \implies -s_n < -M$$

This proves that $\{-s_n\}_{n=1}^{\infty}$ diverges to minus infinity.

28. Suppose $\{s_n\}_{n=1}^{\infty}$ converges to 0. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ converges to 0.

    Ans:

    $\{s_n\}_{n=1}^{\infty}$ converges to 0

    $\implies$ for every $\epsilon > 0$, there is $N \in I$ such that $(n \geq N \implies |s_n - 0| < \epsilon)$

    $\implies$ for every $\epsilon > 0$, there is $N \in I$ such that $(n \geq N \implies |s_n| < \epsilon)$

    $\implies$ for every $\epsilon > 0$, there is $N \in I$ such that $(n \geq N \implies |(-1)^n s_n| < \epsilon)$

    $\implies$ for every $\epsilon > 0$, there is $N \in I$ such that $(n \geq N \implies |(-1)^n s_n - 0| < \epsilon)$

    $\implies$ $\{(-1)^n s_n\}_{n=1}^{\infty}$ converges to 0

29. Suppose $\{s_n\}_{n=1}^{\infty}$ converges to $L \neq 0$. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.

    Ans: Take $\{t_n\}_{n=1}^{\infty} = \{(-1)^n s_n\}_{n=1}^{\infty}$. Note that $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_n = L$ as $\{s_{2n}\}_{n=1}^{\infty}$ and $\{s_{2n+1}\}_{n=1}^{\infty}$ are subsequences of convergent sequence $\{s_n\}_{n=1}^{\infty}$. Now,

    $$\lim_{n\to\infty} t_{2n} = \lim_{n\to\infty} ((-1)^{2n} s_{2n}) = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_n = L$$

    and

    $$\lim_{n\to\infty} t_{2n+1} = \lim_{n\to\infty} ((-1)^{2n+1} s_{2n+1}) = \lim_{n\to\infty} -s_{2n+1} = -\lim_{n\to\infty} s_{2n+1} = -\lim_{n\to\infty} s_n = -L$$

    So
    $$\lim_{n\to\infty} t_{2n} = -\lim_{n\to\infty} t_{2n+1}$$

    implying that the sequence $\{t_n\}_{n=1}^{\infty} = \{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.

30. Suppose $\{s_n\}_{n=1}^{\infty}$ diverges to infinity. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.

    Ans: Take $\{t_n\}_{n=1}^{\infty} = \{(-1)^n s_n\}_{n=1}^{\infty}$.Consider

    $\{t_{2n}\}_{n=1}^{\infty} = \{(-1)^{2n} s_{2n}\}_{n=1}^{\infty} = \{s_{2n}\}_{n=1}^{\infty}$ diverges to infinity and

$\{t_{2n+1}\}_{n=1}^{\infty} = \{(-1)^{2n+1}s_{2n+1}\}_{n=1}^{\infty} = \{-s_{2n+1}\}_{n=1}^{\infty} = -\{s_{2n+1}\}_{n=1}^{\infty}$ diverges to minus infinity.

It follows that the sequence $\{t_n\}_{n=1}^{\infty} = \{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.

31. True or false? If a sequence of positive numbers is not bounded then the sequence diverges to infinity.

    Ans: False. For example the sequence $1, 2, 1, 4, 1, 5, 1, 6, 1, 7, \ldots$ is unbounded sequence of positive numbers that does not diverge to infinity. It is oscillating sequence.

32. Give an example of a sequence $\{s_n\}_{n=1}^{\infty}$ which is not bounded but for which $\lim\limits_{n\to\infty} \frac{s_n}{n} = 0$.

    Ans: Take $\{s_n\}_{n=1}^{\infty} = 1, \sqrt{2}, 1, \sqrt{3}, 1, \sqrt{5} \ldots$. Then $\{s_n\}_{n=1}^{\infty}$ is unbounded. Also

    $\lim\limits_{n\to\infty} \frac{s_{2n-1}}{n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

    and

    $\lim\limits_{n\to\infty} \frac{s_{2n}}{n} = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

    implying that $\lim\limits_{n\to\infty} \frac{s_n}{n} = 0$.

33. (Sandwich Theorem) If $\{s_n\}_{n=1}^{\infty}, \{t_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ are sequences of real number such that $s_n \leq t_n \leq v_n$ for all $n \in I$ and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} v_n = L$, prove that $\{v_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} t_n = L$

    Ans: Let $\epsilon > 0$. Since $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} v_n = L$, there are $N_1, N_2 \in I$ such that

    $n \geq N_1 \implies |s_n - L| < \epsilon$ and $n \geq N_2 \implies |v_n - L| < \epsilon$.

    Taking $N = \max\{N_1, N_2\}$ we obtain that

    $n \geq N \implies |s_n - L| < \epsilon$ and $|v_n - L| < \epsilon$.

    This implies that

    $n \geq N \implies L - \epsilon < s_n$ and $v_n < L + \epsilon$

    which further implies that

    $n \geq N \implies L - \epsilon < t_n$ and $t_n < L + \epsilon \quad (\because s_n \leq t_n \leq v_n)$

    So

$n \geq N \implies |t_n - L| < \epsilon$

This proves that $\lim\limits_{n\to\infty} t_n = L$.

34. If $s_n = \frac{k^n}{n!}$ where k is positive integer, show that $\lim\limits_{n\to\infty} s_n = 0$. In particular show that $\lim\limits_{n\to\infty} \frac{5^n}{n!} = 0$ and $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$.

Ans: For $n \geq k$, consider

$\frac{s_{n+1}}{s_n} = \frac{k^{(n+1)}}{(n+1)!} \times \frac{n!}{k^n} = \frac{k}{n+1} \leq 1$

Thus

$n \geq k \implies \frac{s_{n+1}}{s_n} \leq 1$

$\implies (n \geq k \implies s_{n+1} \leq s_n)$

$\implies \{s_n\}_{n=1}^{\infty}$ is nonincreasing

Also $\{s_n\}_{n=1}^{\infty}$ is bounded below by 0. Therefore $\{s_n\}_{n=1}^{\infty}$ is convergent. Let $\lim\limits_{n\to\infty} s_n = L$. Consider

$n \geq k \implies 0 \leq \frac{s_{n+1}}{s_n} \leq \frac{k}{n+1}$

$\implies (n \geq k \implies 0 \leq s_{n+1} \leq \frac{k}{n+1} s_n)$

$\implies 0 \leq \lim\limits_{n\to\infty} s_{n+1} \leq \lim\limits_{n\to\infty} (\frac{k}{n+1} s_n)$

$\implies 0 \leq L \leq \lim\limits_{n\to\infty} \frac{k}{n+1} \lim\limits_{n\to\infty} s_n$

$\implies 0 \leq L \leq 0.L$

$\implies 0 \leq L \leq 0$

$\implies L = 0$

$\implies \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{k^n}{n!} = 0$

In particular, taking $k = 5$ and $k = 10$, we get $\lim\limits_{n\to\infty} \frac{5^n}{n!} = 0$ and $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$.

35. Prove that if $\lim\limits_{n\to\infty} \frac{s_n}{n} = L \neq 0$, then $\{s_n\}_{n=1}^{\infty}$ is not bounded.

Ans: Suppose on contrary that $\{s_n\}_{n=1}^{\infty}$ is bounded. Then there is $k > 0$ such that $|s_n| \leq k$ for all $n \in I$. Consider

$|s_n| \leq k$

$\implies -k \leq s_n \leq k$

$\implies \frac{-k}{n} \leq \frac{s_n}{n} \leq \frac{k}{n}$

$\implies \lim\limits_{n\to\infty} \frac{-k}{n} \leq \lim\limits_{n\to\infty} \frac{s_n}{n} \leq \lim\limits_{n\to\infty} \frac{k}{n}$

$\implies 0 \leq \lim\limits_{n\to\infty} \frac{s_n}{n} \leq 0$

$\implies \lim\limits_{n\to\infty} \frac{s_n}{n} = 0$

This is a contradiction. Hence $\{s_n\}_{n=1}^{\infty}$ is not bounded.

36. If $\{s_n\}_{n=1}^{\infty}$ is bounded sequence of real numbers, and $\{t_n\}_{n=1}^{\infty}$ converges to 0, prove that $\{s_nt_n\}_{n=1}^{\infty}$ converges to 0.

Ans: As $\{s_n\}_{n=1}^{\infty}$ is bounded, there is a constant $k > 0$ such that $|s_n| \leq k$ for all $n \in I$. Consider

$|s_n| \leq k$

$\implies -k \leq s_n \leq k$

$\implies -kt_n \leq s_nt_n \leq kt_n$

$\implies \lim\limits_{n\to\infty}(-kt_n) \leq \lim\limits_{n\to\infty}(s_nt_n) \leq \lim\limits_{n\to\infty}(kt_n)$

$\implies -k \lim\limits_{n\to\infty} t_n \leq \lim\limits_{n\to\infty}(s_nt_n) \leq k \lim\limits_{n\to\infty} t_n$

$\implies 0 \leq \lim\limits_{n\to\infty}(s_nt_n) \leq 0$

$\implies \lim\limits_{n\to\infty}(s_nt_n) = 0$

37. Let $s_1 = \sqrt{2}$ and let $s_{n+1} = \sqrt{2}\sqrt{s_n}$ for $n \geq 2$.

a) Prove by induction that $s_n \leq \sqrt{2}$ for all n .

b) Prove that $s_{n+1} \geq s_n$ for all n.

c) Prove that $\{s_n\}_{n=1}^{\infty}$ is convergent.

d) Prove that $\lim\limits_{n\to\infty} s_n = 2$.

Ans: a) Let

$P(n) : s_n \leq \sqrt{2}$

As $s_1 = \sqrt{2} \leq 2$, $P(1)$ is true. Let $P(k)$ be true i.e. $s_k \leq 2$. This implies that

$\sqrt{s_k} \leq \sqrt{2}$

$\implies \sqrt{2}\sqrt{s_k} \leq \sqrt{2}\sqrt{2}$

$\implies s_{k+1} \leq 2$

$\implies P(k+1)$ is true

Thus $P(k)$ is true implies $P(k+1)$ is true. So by the principal of mathematical induction $P(n)$ is true for all $n \in I$ i.e. $s_n \leq \sqrt{2}$ for all $n \in I$.

b) From a), $s_n \leq \sqrt{2}$ for all $n \in I$. This implies that

$\sqrt{2} \geq s_n$ for all $n \in I$

$\implies \sqrt{2}\sqrt{s_n} \geq s_n$ for all $n \in I$

$\implies s_{n+1} \geq s_n$ for all $n \in I$

c) From a) and b), $\{s_n\}_{n=1}^{\infty}$ is non-decreasing and upper bounded. Hence $\{s_n\}_{n=1}^{\infty}$ is convergent.

d) $\{s_{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty}$. Hence

$\lim\limits_{n\to\infty} s_{n+1} = \lim\limits_{n\to\infty} s_n = L \quad \text{(suppose)}$

$\implies \lim\limits_{n\to\infty} (\sqrt{2}\sqrt{s_n}) = L$

$\implies \sqrt{2} \lim\limits_{n\to\infty} s_n = L$

$\implies \sqrt{2}\sqrt{L} = L$

$\implies \sqrt{2} = \sqrt{L}$

$\implies 2 = L$

$\implies L = \sqrt{2}$

$\implies \lim\limits_{n\to\infty} s_n = 2$

38. Suppose $s_1 > s_2 > 0$ and let $s_{n+1} = \frac{1}{2}(s_n + s_{n-1})$ for all $n \geq 2$. Prove that

a) $s_1, s_3, s_5, \ldots$ is nonincreasing

b) $s_2, s_4, s_6, \ldots$ is nondecreasing

c) $\{s_n\}_{n=1}^{\infty}$ is convergent.

Ans: Since $s_1 > s_2$

$s_3 = \frac{1}{2}(s_2 + s_1) < \frac{1}{2}(s_1 + s_1) = s_1$ and

$s_3 = \frac{1}{2}(s_1 + s_2) > \frac{1}{2}(s_2 + s_2) = s_2$

Thus

$s_3 < s_1$ and $s_3 > s_2$

This implies that

$s_4 = \frac{1}{2}(s_3 + s_2) > \frac{1}{2}(s_2 + s_2) = s_2$ and

$s_4 = \frac{1}{2}(s_3 + s_2) < \frac{1}{2}(s_3 + s_3) = s_3$ Thus

$s_4 > s_2$ and $s_4 < s_3$

This implies that

$s_5 = \frac{1}{2}(s_4 + s_3) < \frac{1}{2}(s_3 + s_3) = s_3$ and

$s_5 = \frac{1}{2}(s_4 + s_3) > \frac{1}{2}(s_4 + s_4) = s_4$

Thus

$s_5 < s_3$ and $s_5 > s_4$

This implies that

$s_6 = \frac{1}{2}(s_5 + s_4) > \frac{1}{2}(s_4 + s_4) = s_4$ and

$s_6 = \frac{1}{2}(s_5 + s_4) < \frac{1}{2}(s_5 + s_5) = s_6$

Thus

$s_6 > s_4$ and $s_6 < s_5$

Continuing in this way (by induction) we get that

$s_1 > s_3 > s_5 > s_7 \ldots > 0$ and $s_2 < s_4 < s_6 < s_8 \ldots < s_1$

So $\{s_{2n-1}\}_{n=1}^{\infty}$ is nonincreasing and bounded below by 0. Also $\{s_{2n}\}_{n=1}^{\infty}$ is nondecreasing and bounded above by s_1. Therefore $\{s_{2n-1}\}_{n=1}^{\infty}$ and $\{s_{2n}\}_{n=1}^{\infty}$ are convergent. Let $\lim\limits_{n\to\infty} s_{2n-1} = L$ and $\lim\limits_{n\to\infty} s_{2n} = M$. As $s_{2n+2} = \frac{1}{2}(s_{2n+1} + s_{2n})$

$\lim\limits_{n\to\infty} s_{2n+2} = \frac{1}{2}(\lim\limits_{n\to\infty} s_{2n+1} + \lim\limits_{n\to\infty} s_{2n})$

$\implies L = \frac{1}{2}(M + L)$

$\implies L = M$

$\implies \lim\limits_{n\to\infty} s_{2n-1} = \lim\limits_{n\to\infty} s_2 n$

$\implies \lim\limits_{n\to\infty} s_n = L$

$\implies \{s_n\}_{n=1}^{\infty}$ is convergent and converges to L

39. Give an example of sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ for which as $n \to \infty$,
a) $s_n \to \infty, t_n \to \infty, s_n + t_n \to \infty$

b) $s_n \to \infty, t_n \to \infty, s_n - t_n \to 7$

Ans: a) Take $\{s_n\}_{n=1}^{\infty} = \{t_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$

b) Take $\{s_n\}_{n=1}^{\infty} = \{n + 7\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$

40. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a divergent sequence of real numbers and $c \in \mathbb{R}, c \neq 0$. Prove that $\{cs_n\}_{n=1}^{\infty}$ diverges.

Ans: $\{cs_n\}_{n=1}^{\infty}$ is convergent

$\implies \lim\limits_{n\to\infty} (cs_n)$ exists

$\implies \frac{1}{c} \lim\limits_{n\to\infty} (cs_n)$ exists

$\implies \lim\limits_{n\to\infty} (\frac{1}{c} cs_n)$ exists

$\implies \lim\limits_{n\to\infty} s_n$ exists

$\implies \{s_n\}_{n=1}^{\infty}$ is convergent

Thus

$\{cs_n\}_{n=1}^{\infty}$ is convergent $\implies$ $\{s_n\}_{n=1}^{\infty}$ is convergent

which is logically equivalent to

$\{s_n\}_{n=1}^{\infty}$ is divergent $\implies$ $\{cs_n\}_{n=1}^{\infty}$ is divergent

41. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and for each $n \in I$, let
$s_n = a_1 + a_2 + \cdots + a_n$
$t_n = |a_1| + |a_2| + \cdots + |a_n|$
Prove that if $\{t_n\}_{n=1}^{\infty}$ is a Cauchy sequence then so is $\{s_n\}_{n=1}^{\infty}$.

Ans: $\{t_n\}_{n=1}^{\infty}$ is a Cauchy sequence

$\implies \forall \epsilon > 0, \exists N \in I$ such that $(n > m \geq N \implies |t_n - t_m| < \epsilon)$

$\implies \forall \epsilon > 0, \exists N \in I$ such that $(n > m \geq N \implies ||a_m| + |a_{m+1}| + \cdots + |a_n|| < \epsilon)$

$\implies \forall \epsilon > 0, \exists N \in I$ such that $(n > m \geq N \implies |a_m| + |a_{m+1}| + \cdots + |a_n| < \epsilon)$

$\implies \forall \epsilon > 0, \exists N \in I$ such that $(n > m \geq N \implies |a_m + a_{m+1} + \cdots + a_n| < \epsilon)$

($\because |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ by triangle inequality)

$\implies \forall \epsilon > 0, \exists N \in I$ such that $(n > m \geq N \implies |s_n - s_m| < \epsilon)$

$\implies \{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence

Chapter 3: Series of Real Numbers

1. Prove that if $a_1 + a_2 + \cdots$ converges to s, then $a_2 + a_3 + \cdots$ converges to $s - a_1$.

   Ans: Take
   $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = a_2 + a_3 + \cdots a_n$. Consider

   $a_2 + a_3 + \cdots$

   $= \sum_{n=2}^{\infty} a_n$

   $= \lim_{n\to\infty} t_n$

   $= \lim_{n\to\infty} (s_n - a_1)$

   $= \lim_{n\to\infty} s_n - a_1$

   $= \sum_{n=1}^{\infty} a_n - a_1$

   $= s - a_1$

2. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Also find its value.

   Ans: Take $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{(n+1)})$ and let $s_n = a_1 + a_2 + \cdots + a_n$. Consider

   $\sum_{n=1}^{\infty} a_n$

   $= \lim_{n\to\infty} s_n$

   $= \lim_{n\to\infty} [(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots (\frac{1}{n} - \frac{1}{(n+1)})]$

   $= \lim_{n\to\infty} (1 - \frac{1}{(n+1)})$

   $= 1 - 0$

   $= 1$

3. For what values of x does the series $(1 - x) + (x - x^2) + (x^2 - x^3) + \cdots$ converge ?

   Ans: Take $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (x^{n-1} - x^n)$ and let $s_n = a_1 + a_2 + \cdots + a_n$. Consider

   $(1 - x) + (x - x^2) + (x^2 - x^3) + \cdots$

   $= \sum_{n=1}^{\infty} a_n$

   $= \lim_{n\to\infty} s_n$

$= (1-x)+(x-x^2)+(x^2-x^3)+\cdots+(x^{n-1}-x^n)$

$= \lim\limits_{n\to\infty}(1-x^n)$

The limit $\lim\limits_{n\to\infty} x^n$ exists only when $-1 < x \leq 1$. So the series $(1-x)+(x-x^2)+(x^2-x^3)+\cdots$ converges for $x \in (-1, 1]$

4. Prove that the series $(a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\cdots$ converge if and only if the sequence$\{a_n\}_{n=1}^{\infty}$ converges.

   Ans:

   $(a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\cdots$ converges

   $\iff \lim\limits_{n\to\infty}[(a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\cdots+(a_(n-1)+a_n)]$ exists

   $\iff \lim\limits_{n\to\infty}(1-a_n)$ exists

   $\iff \lim\limits_{n\to\infty} a_n$ exists

   $\iff \{a_n\}_{n=1}^{\infty}$ converges

5. Does the series $\sum\limits_{n=1}^{\infty}\log(1+\frac{1}{n})$ converge or diverge?

   Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}\log(1+\frac{1}{n}) = \sum\limits_{n=1}^{\infty}\log(\frac{n+1}{n}) = \sum\limits_{n=1}^{\infty}(\log(n+1)-\log n)$ and let $s_n = a_1+a_2+\cdots a_n$. Consider

   $\lim\limits_{n\to\infty} s_n$

   $= \lim\limits_{n\to\infty}[(\log 2-\log 1)+(\log 3-\log 2)+(\log 4-\log 3)+\cdots+\cdots(\log(n+1)-\log n)]$

   $= \lim\limits_{n\to\infty}\log(n+1)$

   which does not exist. So the series $\sum\limits_{n=1}^{\infty}\log(1+\frac{1}{n})$ diverges.

6. Prove that for any $a, b \in \mathbb{R}$, the series $a+(a+b)+(a+2b)+(a+3b)+\cdots$ diverges unless $a = b = 0$.

   Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}(a+(n-1)b)$ and let $s_n = a_1+a_2+\cdots a_n$. Consider

   $\lim\limits_{n\to\infty} s_n$

   $= \lim\limits_{n\to\infty}[a+(a+b)+(a+2b)+(a+3b)+\cdots+(a+(n-1)b)]$

   $= \lim\limits_{n\to\infty}[na+\frac{n(n-1)}{2}b]$

$= \lim_{n\to\infty} n(a + \frac{(n-1)}{2}b)$

which exists only when $a = b = 0$.

7. Show that $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$ there exists $N \in I$ such that $n > m \geq N \implies |\sum_{k=m+1}^{n} a_k| < \epsilon$.

   Ans: Let $s_n = a_1 + a_2 + \cdots a_n$. Consider

   $\sum_{k=1}^{\infty} a_k$ converges

   $\iff$ $\{s_n\}_{n=1}^{\infty}$ converges

   $\iff$ $\{s_n\}_{n=1}^{\infty}$ is Cauchy

   $\iff$ (for every $\epsilon > 0$, there is $N \in I$ such that $n > m \geq N \implies |s_m - s_n| < \epsilon$)

   $\iff$ (for every $\epsilon > 0$, there is $N \in I$ such that $n > m \geq N \implies |\sum_{k=m+1}^{n} a_k| < \epsilon$)

8. Prove that if $a_1+a_2+a_3+\cdots$ converges to A, then $\frac{1}{2}(a_1+a_2)+\frac{1}{2}(a_2+a_3)+\frac{1}{2}(a_3+a_4)\cdots$ converges. What is the sum of second series?

   Ans: Take $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2}(a_n + a_{n+1})$ and let $t_n = b_1 + b_2 + \cdots b_n$. Consider

   $\frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_2 + a_3) + \frac{1}{2}(a_3 + a_4)\cdots$

   $= \sum_{n=1}^{\infty} b_n$

   $= \lim_{n\to\infty} t_n$

   $= \lim_{n\to\infty} (b_1 + b_2 + \cdots b_n)$

   $= \lim_{n\to\infty} (\frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_2 + a_3) + \frac{1}{2}(a_3 + a_4)\cdots + \frac{1}{2}(a_n + a_{n+1}))$

   $= \lim_{n\to\infty} \frac{1}{2}(a_1 + a_{n+1}) + \lim_{n\to\infty} (a_2 + a_3 + \cdots + a_n)$

   $= \frac{1}{2}(a_1 + 0) + \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) - a_1$

   $= \frac{1}{2}a_1 + A - a_1$

   $= A - \frac{1}{2}a_1$

9. Do the series $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ and $\sum_{n=1}^{\infty} \frac{n+1}{10^{10}(n+2)}$ converge or diverge?

Ans: $\lim\limits_{n\to\infty} \frac{n+1}{n+2} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \lim\limits_{n\to\infty} \frac{1+0}{1+0} = 1 \neq 0$ and

$\lim\limits_{n\to\infty} \frac{n+1}{10^{10}(n+2)} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{10^{10}(1+\frac{2}{n})} = \frac{1+0}{10^{10}+0} = \frac{1}{10^{10}} \neq 0.$

So both the series diverge.

10. Show that if $a_1+a_2+a_3+\cdots$ converges to L, then so does $a_1+0+a_2+0+a_3+0+\cdots$.

Ans: Let $b_n = a_{\frac{n-1}{2}}$ for $n = 1, 3, 5, \cdots$ and $b_n = 0$ for $n = 2, 4, 6, \cdots$. Take $t_n = b_1 + b_2 + b_3 + \cdots + b_n$. Then

$\lim\limits_{n\to\infty} t_{2n}$

$= \lim\limits_{n\to\infty} (b_1 + b_2 + b_3 + \cdots + b_{2n})$

$= \lim\limits_{n\to\infty} a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots + a_n + 0$

$= \lim\limits_{n\to\infty} (a_1 + a_2 + a_3 + \cdots + a_n)$

$= \sum\limits_{n=1}^{\infty} a_n$

$= L$

and

$\lim\limits_{n\to\infty} t_{2n+1}$

$= \lim\limits_{n\to\infty} (b_1 + b_2 + b_3 + \cdots + b_{2n+1})$

$= \lim\limits_{n\to\infty} a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots + 0 + a_n$

$= \lim\limits_{n\to\infty} (a_1 + a_2 + a_3 + \cdots + a_n)$

$= \sum\limits_{n=1}^{\infty} a_n$

$= L$

Therefore $\lim\limits_{n\to\infty} t_n = L$. Now

$a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots$

$= \sum\limits_{n=1}^{\infty} b_n$

$= \lim\limits_{n\to\infty} t_n$

$= L$

11. Prove that if $\sum\limits_{n=1}^{\infty} a_n$ converges and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty}(a_n+b_n)$ diverges.

Ans: Suppose on contrary that $\sum\limits_{n=1}^{\infty}(a_n+b_n)$ converges. As $\sum\limits_{n=1}^{\infty} a_n$ converges, $\sum\limits_{n=1}^{\infty}[(a_n+b_n)-a_n]=\sum\limits_{n=1}^{\infty} b_n$ also converges. This is a contradiction. Hence $\sum\limits_{n=1}^{\infty}(a_n+b_n)$ can't converge.

12. Give an example of a series $\sum\limits_{n=1}^{\infty} a_n$ such that $(a_1+a_2)+(a_3+a_4)+\cdots$ converges but $a_1+a_2+a_3+a_4+\cdots$ diverges.

Ans: Take $\sum\limits_{n=1}^{\infty} a_n=\sum\limits_{n=1}^{\infty}(-1)^n$. Let $s_n=a_1+a_2+a_3+\cdots+a_n$. Then $\lim\limits_{n\to\infty} s_{2n}=0$ and $\lim\limits_{n\to\infty} s_{2n-1}=-1$. Therefore $\lim\limits_{n\to\infty} s_n$ does not exist and hence $\sum\limits_{n=1}^{\infty} a_n=a_1+a_2+a_3+a_4+\cdots$ diverges.

Take $\sum\limits_{n=1}^{\infty} b_n=\sum\limits_{n=1}^{\infty}(a_{2n-1}+a_{2n})$. Let $t_n=b_1+b_2+b_3+\cdots+b_n$. Consider

t_n

$=b_1+b_2+b_3+\cdots+b_n$

$=(a_1+a_2)+(a_3+a_4)+(a_5+a_6)+\cdots+(a_{2n-1}+a_{2n})$

$=(-1+1)+(-1+1)+(-1+1)+\cdots+(-1+1)$

$=0+0+0+\cdots+0$

$=0$

Therefore $\lim\limits_{n\to\infty} t_n=0$ and hence $\sum\limits_{n=1}^{\infty} b_n=\sum\limits_{n=1}^{\infty}(a_{2n-1}+a_{2n})=(a_1+a_2)+(a_3+a_4)+\cdots$ converges to 0.

13. If $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, and if $\{a_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$, prove that $\sum\limits_{i=1}^{\infty} a_{n_i}$ converges.

Ans: Take $s_n=a_1+a_2+\cdots+a_n$. Then $\{s_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty}$. Consider

$\sum\limits_{n=1}^{\infty} a_n$ converges

$\implies \{s_n\}_{n=1}^{\infty}$ converges

$\implies \{s_{n_i}\}_{i=1}^{\infty}$ converges ($\because$ a subsequence of convergent sequence is convergent)

$\implies \sum\limits_{i=1}^{\infty} a_{n_i}$ converges

14. Prove that $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ converges.

Ans: Let $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n!}$ and let $s_n = a_1 + a_2 + \cdots + a_n$. Consider

$n \geq 2$

$\implies 1.2.3\ldots n \geq \underbrace{2.2\ldots 2}_{(n-1)\ terms}$

$\implies n! \geq 2^{n-1}$

Thus

$n \geq 2 \implies n! \geq 2^{n-1}$

so that

$n > 2 \implies \frac{1}{n!} \leq \frac{1}{2^{n-1}}$

and hence

$n > 2 \implies a_n \leq \frac{1}{2^{n-1}}$

Now,

$s_n = a_1 + a_2 + \cdots + a_n$

$\leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$

$= \frac{1.(1-2^{-n})}{1-2^{-1}}$

$= 2(1-2^{-n})$

≤ 2

Thus the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the non-negative series $\sum\limits_{n=1}^{\infty} a_n$ is bounded above. Hence the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

15. Prove that $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$ converges.

Ans: Let $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{(2n-2)!}$ and let $s_n = a_1 + a_2 + \cdots + a_n$. Consider

$n \geq 2$

$\implies 1.2.3\ldots n \geq \underbrace{2.2\ldots 2}_{(n-1)\ terms}$

$\implies n! \geq 2^{n-1}$

Thus

$n \geq 2 \implies n! \geq 2^{n-1}$

This implies that

$n > 2 \implies (2n-2)! \geq 2^{2n-3}$

so that

$n > 2 \implies \frac{1}{(2n-2)!} \leq \frac{1}{2^{2n-3}}$

and hence

$n > 2 \implies a_n \leq \frac{1}{2^{2n-3}}$

Now,

$s_n = a_1 + a_2 + \cdots + a_n$

$\leq 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-3}}$

$= 1 + \frac{\frac{1}{2}(1-2^{-2n})}{1-2^{-2}}$

$= 1 + \frac{2}{3}(1-2^{-2n})$

$\leq 1 + \frac{2}{3}$

$= \frac{5}{3}$

Thus the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the non-negative series $\sum\limits_{n=1}^{\infty} a_n$ is bounded above. Hence the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{(2n-2)!}$ is convergent.

16. If $0 \leq a_n \leq 1 (n \geq 0)$ and if $0 \leq x < 1$, then prove that $\sum\limits_{n=1}^{\infty} a_n x^n$ converges, and that its sum is not greater than $\frac{1}{1-x}$.

Ans: Let $s_n = a_1x + a_2x^2 + \cdots + a_nx^n$. Consider

$a_1x + a_2x^2 + \cdots + a_nx^n$

$\leq 1 + x + x^2 + \cdots + x^n \quad (\because 0 \leq a_n \leq 1 \text{ for } n \geq 0)$

$= \frac{1.(1-x^{n+1})}{1-x}$

$= \frac{1-x^{(n+1)}}{1-x}$

$\leq \frac{1}{1-x} \quad (\because 0 \leq x < 1)$

Thus

$s_n \leq \frac{1}{1-x}$ for all $n \in I$

This implies that $\sum\limits_{n=1}^{\infty} a_n x^n$ converges and

$\sum\limits_{n=1}^{\infty} a_n x^n = \lim\limits_{n\to\infty} s_n \leq \frac{1}{1-x}$

17. Show that the series $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

    Ans: Take $a_n = \frac{1}{n}$. Then

    $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots$

    $\implies a_1 > a_2 > a_3 > a_4 > \cdots$

    and

    $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

    It follows that $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

18. Show that the series $\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$ is convergent.

    Ans: Take $a_n = \frac{1}{n!}$. Then

    $\frac{1}{1!} > \frac{1}{2!} > \frac{1}{3!} > \frac{1}{4!} > \cdots$

    $\implies a_1 > a_2 > a_3 > a_4 > \cdots$

    and

    $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n!} = 0$

    It follows that $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$ is convergent.

19. Show that the series $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots$ is convergent.

Ans: Take $a_n = \frac{1}{(n-1)!}$. Then

$\frac{1}{0!} > \frac{1}{1!} > \frac{1}{2!} > \frac{1}{3!} > \frac{1}{4!} > \cdots$

$\implies a_1 > a_2 > a_3 > a_4 > a_5 > \cdots$

and

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{(n-1)!} = 0$

It follows that $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots$ is convergent.

20. For what values of p does the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ converge?

Ans: Take $a_n = \frac{1}{n^p}$. Note that $\lim\limits_{n\to\infty} a_n = 0$ only when $p > 0$. Also if $p > 0$, then

$n^p < (n+1)^p$ implying that $\frac{1}{n^p} > \frac{1}{(n+1)^p}$ i.e. $a_n > a_{n+1}$. Thus

$a_1 > a_2 > a_3 > \cdots$ and $\lim\limits_{n\to\infty} a_n = 0$ only when $p > 0$.

It follows that $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = \frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ converges only when $p > 0$.

21. If x is not an integer, prove that $\frac{1}{(x+1)} - \frac{1}{(x+2)} + \frac{1}{(x+3)} + \cdots$ converges.

Ans: Take $a_n = \frac{1}{x+n}$. By Archimedian property there is $N \in I$ such that $N > -x$ i.e. $x + N > 0$ Now, for $n \geq N$

$1 > 0$

$\implies n + 1 > n$

$\implies x + (n+1) > x + n$

$\implies \frac{1}{x+n+1} < \frac{1}{x+n} \qquad (\because \frac{1}{x+n} > 0 \text{ as } x + n \geq x + N > 0)$

$\implies a_{n+1} < a_n$

Thus

$n \geq N \implies a_{n+1} < a_n$

implying that

$a_N > a_{N+1} > a_{N+2} > \ldots$

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{x+n} = 0.$

So $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = \frac{1}{(x+1)} - \frac{1}{(x+2)} + \frac{1}{(x+3)} + \cdots$ converges.

22. Prove that $2 - 2^{\frac{1}{2}} + 2^{\frac{1}{3}} - 2^{\frac{1}{4}} + \cdots$ diverges.

    Ans: Take $a_n = (-1)^{n+1}2^{\frac{1}{n}}$. Consider

    $\lim\limits_{n\to\infty} a_{2n+1}$

    $= \lim\limits_{n\to\infty} \frac{2}{\frac{1}{2n+1}}$

    $= 2^{\lim\limits_{n\to\infty}\frac{1}{2n+1}}$

    $= 2^0$

    $= 1$

    So $\lim\limits_{n\to\infty} a_{2n+1} \neq 0$ implying that $\lim\limits_{n\to\infty} a_n \neq 0$. Therefore $\sum\limits_{n=1}^{\infty} a_n = 2 - 2^{\frac{1}{2}} + 2^{\frac{1}{3}} - 2^{\frac{1}{4}} + \cdots$ diverges.

23. Prove that $(1-2) - (1-2^{\frac{1}{2}}) + (1-2^{\frac{1}{3}}) - (1-2^{\frac{1}{4}}) + \cdots$ converges.

    Ans: Take $a_n = 2^{\frac{1}{n}} - 1$. Consider

    $n < n+1$

    $\implies \frac{1}{n} > \frac{1}{n+1}$

    $\implies 2^{\frac{1}{n}} > 2^{\frac{1}{n+1}}$

    $\implies 2^{\frac{1}{n}} - 1 > 2^{\frac{1}{n+1}} - 1$

    $\implies a_n > a_{n+1}$

    This holds for all $n \in I$. So

    $a_1 > a_2 > a_3 \ldots$

    Also

    $\lim\limits_{n\to\infty} a_n = (\lim\limits_{n\to\infty} 2^{\frac{1}{n}} - 1) = 2^{\lim\limits_{n\to\infty}\frac{1}{n}} - 1 = 2^0 - 1 = 1 - 1 = 0$

    So $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent implying the convergence of $(-1)\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n = \sum\limits_{n=1}^{\infty}(-1)^{n+2}a_n = (1-2) - (1-2^{\frac{1}{2}}) + (1-2^{\frac{1}{3}}) - (1-2^{\frac{1}{4}}) + \cdots$

24. Show that $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$ diverges.

    Ans: Take $a_n = (-1)^{n+1}\frac{n}{2n+1}$. Consider

    $\lim\limits_{n\to\infty} a_{2n+1}$

    $= \lim\limits_{n\to\infty} \frac{2n+1}{4n+1}$

    $= \lim\limits_{n\to\infty} \frac{2+\frac{1}{n}}{4+\frac{1}{n}}$

    $= \frac{2+0}{4+0}$

    $= \frac{1}{2}$

    So $\lim\limits_{n\to\infty} a_{2n+1} \neq 0$ implying that $\lim\limits_{n\to\infty} a_n \neq 0$. Therefore $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$ diverges.

25. Can a series of nonnegative numbers converge conditionally?

    Ans: No.

    $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series of non-negative real numbers.

    $\implies \sum\limits_{n=1}^{\infty} |a_n|$ is convergent $\quad (\because |a_n| = a_n \text{ as } a_n > 0)$

    $\implies \sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

26. Prove that if $\sum\limits_{n=1}^{\infty} |a_n| < \infty$, then $|\sum\limits_{n=1}^{\infty} a_n| \leq \sum\limits_{n=1}^{\infty} |a_n|$.

    Ans: Take $s_n = |a_1|+|a_2|+\cdots+|a_n|$ and $t_n = |a_1+a_2+\cdots+a_n|$. By triangle inequality

    $t_n = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = s_n$

    This implies that

    $\lim\limits_{n\to\infty} t_n \leq \lim\limits_{n\to\infty} s_n$

    $\implies \lim\limits_{n\to\infty} (|a_1 + a_2 + \cdots + a_n|) \leq \lim\limits_{n\to\infty} (|a_1| + |a_2| + \cdots + |a_n|)$

    $\implies |\lim\limits_{n\to\infty} (a_1 + a_2 + \cdots + a_n)| \leq \sum\limits_{n=1}^{\infty} |a_n| \qquad (\because \lim\limits_{n\to\infty} |x_n| = |\lim\limits_{n\to\infty} x_n|)$

    $\implies |\sum\limits_{n=1}^{\infty} a_n| \leq \sum\limits_{n=1}^{\infty} |a_n|$

27. **(Geometric Series)** Show that $\sum\limits_{n=0}^{\infty} x^n$ converges if and only if $-1 < x < 1$.

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} x^n$ and let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$

I Case: $x < -1$: In this case, $\{a_n\}_{n=1}^{\infty}$ is non-increasing and not-bounded. Hence it diverges to $-\infty$ and so $\lim\limits_{n\to\infty} a_n \neq 0$. Therefore, $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

II Case: $x = -1$ In this case $\{s_{2n}\}_{n=1}^{\infty}$ converges to 0 and $\{s_{2n+1}\}_{n=1}^{\infty}$ converges to 1. So the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent. This implies the divergence of the series $\sum\limits_{n=1}^{\infty} a_n$.

III Case: $-1 < x < 1$: In this case,

s_n

$= 1 + x + x^2 + \cdots + x^n$

$= \frac{1-x^n}{1-x}$

So

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x} \qquad (\because \lim\limits_{n\to\infty} x^n = 0 \text{ as } -1 < x < 1).$

Thus the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent and converges to $\frac{x}{1-x}$

IV Case: $x \geq 1$: In this case, $\{a_n\}_{n=1}^{\infty}$ is non-decreasing and not-bounded. Hence it diverges to ∞ and so $\lim\limits_{n\to\infty} a_n \neq 0$. Therefore, $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

$$\boxed{\text{The geometric series } \sum_{n=1}^{\infty} x^n \text{ converges } \iff -1 < x < 1}$$

28. (**p-Series**)Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Ans:

I Case: $p > 1$: Consider

$$\underbrace{\frac{1}{(2^{j-1})^p} + \frac{1}{(2^{j-1}+1)^p} + \frac{1}{(2^{j-1}+2)^p} + \cdots + \frac{1}{(2^j-1)^p}}_{2^{j-1}\ terms} \leq 2^{j-1}\frac{1}{(2^{j-1})^p}$$

(each term less than or equal to first term)

With this observation

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

$= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

$= 1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}) + \cdots + (\frac{1}{(2^{j-1})^p} + \frac{1}{(2^{j-1}+1)^p} + \frac{1}{(2^{j-1}+2)^p} + \cdots + \frac{1}{(2^j-1)^p}) + \cdots$

$\leq 1 + 2.\frac{1}{2^p} + 2^2.\frac{1}{(2^2)^p} + \cdots + 2^{j-1}\frac{1}{(2^j-1)^p} + \cdots$

Thus

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p} \ll 1 + 2.\frac{1}{2^p} + 2^2.\frac{1}{(2^2)^p} + \cdots + 2^{j-1}\frac{1}{(2^j-1)^p} + \cdots$

The series on right is a geometric series with common ratio $\frac{1}{2^{p-1}} < 1$ ($\because p > 1$). By comparison test, it follows that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

II Case: $p = 1$: Let $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ and take $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. Consider

$s_1 = 1$

$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}$

Continuing in this way, $s_{2^n} > \frac{(n+2)}{2}$.

So the sequence $\{s_n\}_{n=1}^{\infty}$ contains a divergent subsequence and hence it is divergent. This implies the divergence of the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.

III Case: $p < 1$: In this case, $\frac{1}{n} < \frac{1}{n^p}$. So

$\sum\limits_{n=1}^{\infty} \frac{1}{n} \ll \sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

From **II Case**, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent. So by comparison test, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

The $p-$ series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$

29. Show that if $|x| < 1$, then $\sum\limits_{n=1}^{\infty} n^{10000} x^n$ converges absolutely.

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} n^{10000} x^n$. Consider

$\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

$= \lim_{n\to\infty} \left(|\frac{(n+1)^{10000}}{x^{n+1}}| \times |\frac{n^{10000}}{x^n}|\right)$

$= \lim_{n\to\infty} \left((1+\frac{1}{n})^{10000}|x|\right)$

$= (1+0)^{10000}|x|$

$= |x|$

< 1

So by ratio test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^{10000}x^n$ converges absolutely.

30. For any $x > 0$ prove that the series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ and $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converge absolutely.

Ans:

A): Take $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Consider

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

$= \lim_{n\to\infty} \left(|(-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!}| \times |(-1)^n\frac{(2n)!}{x^{2n}}|\right)$

$= \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)}$

$= 0$

< 1

So by ratio test the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ converges absolutely.

B): Take $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Consider

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

$= \lim_{n\to\infty} \left(|(-1)^{n+1}\frac{x^{2n+3}}{(2n+3)!}| \times |(-1)^n\frac{(2n+1)!}{x^{2n+1}}|\right)$

$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)}$

$= 0$

< 1

So by ratio test the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges absolutely.

31. For what values of x does the series $1 + 2x + 3x^2 + 4x^3 + \cdots$ converge?

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1}$. Consider

$\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

$= \lim\limits_{n\to\infty} \frac{(n+1)|x|^n}{n|x|^{n-1}}$

$= \lim\limits_{n\to\infty} \big((1+\frac{1}{n})|x|\big)$

$= (1+0)|x|$

$= |x|$

So by ratio test, the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1}$ converges when $|x| < 1$ (i.e. when $-1 < x < 1$) and diverges when $|x| > 1$ (i.e. when $x < -1$ *or* $x > 1$).

For $x = 1$, the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1} = 1 + 2 + 3 + \cdots$ is divergent as $\lim\limits_{n\to\infty} a_n \neq 0$

and

for $x = -1$, $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1} = 1 - 2 + 3 - \cdots$ is divergent as $\lim\limits_{n\to\infty} a_n \neq 0$

Thus

the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1}$ converges for $-1 < x < 1$

and

the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} nx^{n-1}$ diverges for $x \leq -1$ or $x \geq 1$

32. Test the convergence of $\sum\limits_{n=1}^{\infty} a_n$ where $a_n = (3-e)(3-e^{\frac{1}{2}})(3-e^{\frac{1}{3}})\cdots(3-e^{\frac{1}{n}})$.

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (3-e)(3-e^{\frac{1}{2}})(3-e^{\frac{1}{3}})\cdots(3-e^{\frac{1}{n}})$. Consider

. $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

$= \lim\limits_{n\to\infty} \frac{(3-e)(3-e^{\frac{1}{2}})(3-e^{\frac{1}{3}})\cdots(3-e^{\frac{1}{n}})(3-e^{\frac{1}{n+1}})}{(3-e)(3-e^{\frac{1}{2}})(3-e^{\frac{1}{3}})\cdots(3-e^{\frac{1}{n}})}$

$= \lim\limits_{n\to\infty} (3 - e^{\frac{1}{n+1}})$

$= 3 - e^0$

$= 3 - 1$

$= 2$

> 1

So by ratio test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (3-e)(3-e^{\frac{1}{2}})(3-e^{\frac{1}{3}})\cdots(3-e^{\frac{1}{n}})$ diverges.

33. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and if $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, prove that $\lim\limits_{n\to\infty} a_n = 0$.

Ans:

$\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$

$\implies \sum_{n=1}^{\infty} a_n$ converges absolutely (by ratio test)

$\implies \sum_{n=1}^{\infty} a_n$ converges

$\implies \lim\limits_{n\to\infty} a_n = 0$

34. Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges for all $x \in \mathbb{R}$.

Ans: Take $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^n}$. Consider

$\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} |\frac{x^n}{n^n}|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \frac{|x|}{n}$

$= 0$ for all $x \in \mathbb{R}$

< 1

So by root test the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges for all $x \in \mathbb{R}$.

35. If $\sum_{n=1}^{\infty} |a_n| < \infty$ and if for each $n \in I$, $\frac{|b_{n+1}|}{|b_n|} \leq \frac{|a_{n+1}|}{|a_n|}$, prove that $\sum_{n=1}^{\infty} |a_n| < \infty$.

Ans: Consider

$\frac{|b_n|}{|b_1|}$

$= \frac{|b_n|}{|b_{n-1}|} \times \frac{|b_{n-1}|}{|b_{n-2}|} \times \frac{|b_{n-2}|}{|b_{n-3}|} \times \cdots \times \frac{|b_3|}{|b_2|} \times \frac{|b_2|}{|b_1|}$

$\leq \frac{|a_n|}{|a_{n-1}|} \times \frac{|a_{n-1}|}{|a_{n-2}|} \times \frac{|a_{n-2}|}{|a_{n-3}|} \times \cdots \times \frac{|a_3|}{|a_2|} \times \frac{|a_2|}{|a_1|}$

$= \frac{|a_n|}{|a_1|}$

Thus

$\frac{|b_n|}{|b_1|} \leq \frac{|a_n|}{|a_1|}$ for all $n \in I$

$\implies |b_n| \leq \frac{|b_1|}{|a_1|}|a_n|$ for all $n \in I$

$\implies \sum\limits_{n=1}^{\infty} |b_n| \ll \sum\limits_{n=1}^{\infty} |a_n|$

$\implies \sum\limits_{n=1}^{\infty} |b_n| < \infty$ (by comparison test as $\sum\limits_{n=1}^{\infty} |a_n| < \infty$)

36. Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{1}{(\log n)^n}$

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \frac{1}{(\log n)^n}$. Consider

$\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} |\frac{1}{(\log n)^n}|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \frac{1}{\log n}$

$= 0$

< 1

So by root test the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n^n}$ converges.

37. Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{(1+\frac{1}{n})^{2n}}{e^n}$.

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(1+\frac{1}{n})^{2n}}{e^n}$. Consider

$\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} |\frac{(1+\frac{1}{n})^{2n}}{e^n}|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \frac{(1+\frac{1}{n})^2}{e}$

$= \frac{(1+0)^2}{e}$

$= \frac{1}{e}$

< 1

So by root test the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(1+\frac{1}{n})^{2n}}{e^n}$ converges.

38. For what values of x does the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$. Consider

$\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \left(|\frac{x^n}{n}|^{\frac{1}{n}}\right)$

$= \lim\limits_{n\to\infty} \left(\frac{|x|}{n^{\frac{1}{n}}}\right)$

$= \frac{|x|}{1} \qquad (\because \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1)$

$= |x|$

So by root test the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ converges when $|x| < 1$ (i.e. when $-1 < x < 1$) and diverges when $|x| > 1$ (i.e. when $x < -1$ or $x > 1$).

Also for $x = -1$, the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and for $x = 1$, the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

Thus

the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ is convergent for $-1 \leq x < 1$

and

the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ is divergent for $x < -1 \; or \; x \geq 1$

39. For what values of x does the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^x}$ converge?

Ans: Take $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^x}$ and $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} 2^n a_{2^n}$. Consider

$\sum\limits_{n=1}^{\infty} b_n$

$= \sum\limits_{n=1}^{\infty} 2^n a_{2^n}$

$= \sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^x}$

$= \sum\limits_{n=1}^{\infty} (2^{(1-x)})^n$

This is a geometric series and it converges if and only if $2^{(1-x)} > 1 \iff 1 - x > 0 \iff 1 > x$. By Cauchy's condensation test, $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.

It follows that $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^x}$ converges if and only if $x > 1$.

40. For what values of x does the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^x}$ converge?

Ans: $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^x}$ converges

$\iff \sum\limits_{n=2}^{\infty} 2^n \frac{1}{2^n(\log 2^n)^x}$ converges (by Cauchy's condensation test)

$\iff \sum\limits_{n=2}^{\infty} \frac{1}{(\log 2)^x n^x}$ converges

$\iff \sum\limits_{n=2}^{\infty} \frac{1}{n^x}$ converges

$\iff x > 1$ (p series converges only when $p > 1$)

41. Prove that for any real x the series $\sum\limits_{n=3}^{\infty} \frac{1}{(\log n)^x}$ diverges.

Ans: Take $\sum\limits_{n=3}^{\infty} a_n = \sum\limits_{n=3}^{\infty} \frac{1}{(\log n)^x}$, $\sum\limits_{n=3}^{\infty} b_n = \sum\limits_{n=3}^{\infty} 2^n a_{2^n}$. and $\sum\limits_{n=3}^{\infty} c_n = \frac{2^n}{n^x}$. Consider $\sum\limits_{n=3}^{\infty} b_n = \sum\limits_{n=3}^{\infty} 2^n \frac{1}{(\log 2^n)^x} = \frac{1}{(\log 2)^x} \sum\limits_{n=3}^{\infty} \frac{2^n}{n^x} = \frac{1}{(\log 2)^x} \sum\limits_{n=3}^{\infty} c_n$.

As
$\lim\limits_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim\limits_{n\to\infty} \left(\frac{2^{n+1}}{(n+1)^x} \times \frac{n^x}{2^n}\right) = \lim\limits_{n\to\infty} \left(2\frac{1}{(1+\frac{1}{n})^x}\right) = \frac{2}{(1+0)^x} = 2 > 1$,

by ratio test $\sum\limits_{n=3}^{\infty} c_n$ and hence $\sum\limits_{n=3}^{\infty} b_n$ is divergent.Therefore by Cauchy's condensation test, $\sum\limits_{n=3}^{\infty} a_n = \sum\limits_{n=3}^{\infty} \frac{1}{(\log n)^x}$ diverges.